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UNIQUENESS PROBLEMS OF DIFFERENCE-DIFFERENTIAL POLYNOMIALS SHARING ONE VALUE OF ENTIRE FUNCTIONS

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Abstract. In this paper, we study the uniqueness problems of difference-differential polynomials of entire functions f and g sharing one value with counting multiplicity (CM). The results extend and improve the results of Renukadevi S. Dyavanal and Ashwini M. Hattikal [3].

1. Introduction and main results

A meromorphic function $f(z)$ means meromorphic in the complex plane. If no poles occur, then $f(z)$ reduces to an entire function. We assume that the reader is familiar with the notations and the basic results of Nevanlinna theory of meromorphic functions [7], [14] and [16]. For any nonconstant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure.

Let a be a finite complex plane and k be a positive integer. We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for the zeros of $f(z) - a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the counting function for the zeros of $f(z) - a$ with multiplicity $\geq k$ and $\overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the corresponding one for which multiplicity is not counted. Moreover, we set

$$N_k\left(r, \frac{1}{(f-a)}\right) = \overline{N}\left(r, \frac{1}{(f-a)}\right) + \overline{N}_{(2}\left(r, \frac{1}{(f-a)}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right).$$

In the same way, we can define $N_k(r, f)$.

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Let f and g be non-constant meromorphic functions and a be a complex number. We say f and g share the value a CM, if $f-a$ and $g-a$ have the same zeros with the same multiplicities.

In 1993, Wang and Fang [12, 13] proved the following theorem for transcendental entire functions.

Theorem A. Let $f(z)$ be a transcendental entire function. n and k be two positive integers with $n \geq k+1$, then $[f^n]^{(k)}-1$ has infinitely many zeros.

In 2002, M. L. Fang [4] proved the unicity theorem corresponding to the above result.

Theorem B. Let f and g be two non-constant entire functions, and let $n \geq 11$ be a positive integer with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$, or $f \equiv tg$ for a constant t such that $t^n=1$.

In 2008, X. Y. Zhang, J. F. Chen and W. C. Lin [18] proved the following results on uniqueness of two polynomials sharing a common value.

Theorem C. Let f be a transcendental entire function, let n, k and m be positive integers with $n \geq k+2$, and $P(z)=a_0+a_1z+a_2z^2+...+a_mz^m$, where $a_0, a_1, a_2, ..., a_m$ are complex constants. Then $[f^nP(f)]^{(k)}=1$ has infinitely many solutions.

Theorem D. Let f and g be two non-constant entire functions. Let n, k and m be three positive integers with $n \geq 3m+2k+5$, and $P(z)=a_mz^m+a_{m-1}z^{m-1}+ ... + a_1z+a_0$ or $P(z) \equiv c_0$, where $a_0(\neq 0), a_1, a_2, a_3, ..., a_{m-1}, a_m(\neq 0)$, $c_0(\neq 0)$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then
(1) when $P(z)=a_mz^m+a_{m-1}z^{m-1}+...+a_1z+a_0$, either $f(z) \equiv tg(z)$ for a constant t such that $t^d=1$, where $d=(n+m, ..., n+m-i, ..., n), a_{m-i}(\neq 0)$ for some $i=0,1,2,...,m$, or f and g satisfy the algebraic equation $R(f,\, g) \equiv 0$, where $R(w_1,\, w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+...+a_0)-w_2^n(a_mw_2^m+a_{m-1}w_2^{m-1}+...+a_0)$
(2) when $P(z) \equiv c_0$, either $f(z)=\frac{c_1}{\sqrt[n]{c_0}e^{cz}}$, $g(z)=\frac{c_2}{\sqrt[n]{c_0}e^{-cz}}$, where c_1, c_2 and c are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$, or $f \equiv tg$ for a constant t such that $t^n=1$.

In 2012, L. Kai, L. Xin-ling, C. Ting-bin [8] considered Theorem B for difference-differential polynomials and proved the following results.

Theorem E. Let $f(z)$ be a transcendental entire function of finite order. If $n \geq k+2$, then the difference-differential polynomial $[f^n(z)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem F. Let f and g be transcendental entire functions of finite order, $n \geq 2k+6$ and c is a non-zero complex constant. If $[f^n(z)f(z+c)]^{(k)}$ and $[g^n(z)g(z+c)]^{(k)}$ share the value 1 CM, then either $f(z) = c_1e^{Cz}$, $g(z) = c_2e^{-Cz}$, where c_1, c_2 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+1)C)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

In the same direction J. Zhang [17] investigated the value distribution and uniqueness of difference polynomials of entire functions and obtained the following results.

Theorem G. Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that c is a non-zero complex constant. If $n \geq 2$, then $f^n(z)(f(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.

Theorem H. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero constant and n is an integer. If $n \geq 7$, then $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.

In 2014, R. S. Dyavanal and R. V. Desai [2] extended the results of J. Zhang [17] and proved the following results.

Theorem I. Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that c is a non-zero complex constant and n is an integer. If $n \geq 2$, $k_1 \geq 1$ then $f^n(z)(f(z)-1)^{k_1}f(z+c) - \alpha(z)$ has infinitely many zeros.

Theorem J. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant, $k_1 \geq 1, n \geq k_1 + 6$. If $f^n(z)(f(z)-1)^{k_1}f(z+c)$ and $g^n(z)(g(z)-1)^{k_1}g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv tg(z)$, where $t^{k_1} = 1$.

Recently, Renukadevi S. Dyavanal and Ashwini M. Hattikal [3] considered Theorem C and Theorem D to difference-differential polynomials and extends the above theorems as follows.

Theorem K. Let f be a transcendental entire function. n, k and m be positive integers with $n \geq k+2$ and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0, a_1, a_2, a_3, ..., a_{m-1}, a_m$ are complex constants and $\alpha(z)$ be a small function with respect to $f(z)$. Then $[f^n P(f) f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem L. Let f and g be two non-constant entire functions of finite order. Let n, k and m be three positive integers with $n \geq m + 2k + 6$, 'c' is a non-zero complex constant and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$ or $P(z) \equiv c_0$, where $a_0 (\neq 0), a_1, a_2, a_3, ..., a_{m-1}, a_m (\neq 0)$, $c_0 (\neq 0)$ are complex constants. If $[f^n(z) P(f) f(z+c)]^{(k)}$ and $[g^n(z) P(g) g(z+c)]^{(k)}$ share 1 CM, then
(1) when $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, we get $f(z) \equiv t g(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$ and $i = 0, 1, 2, ..., m$.
(2) when $P(z) \equiv c_0$, either $f(z) = \frac{c_1 e^{Cz}}{\sqrt[n]{c_0}}$, $g(z) = \frac{c_2 e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1, c_2, c_0 and C are constants satisfying $(-1)^k (c_1 c_2)^{n+1} ((n+1)C)^{2k} = (\sqrt[n]{c_0})^2$, or $f \equiv t g$ for a constant t such that $t^{n+1} = 1$.

In this paper, we assume $c_j \in \mathbb{C} \setminus \{0\}\, (j = 1, ..., d)$ are distinct constants, n, m, d, $s_j\ (j = 1, ..., d) \in N_+$. $\lambda = \sum_{j=1}^{d} s_j = s_1 + ... + s_d$. Let

$$F(z) = f^n(z) P(f) \prod_{j=1}^{d} f(z+c_j)^{s_j}, \; G(z) = g^n(z) P(g) \prod_{j=1}^{d} g(z+c_j)^{s_j}. \tag{1.1}$$

Where $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_0$ is a nonzero polynomial of degree m, let $\Gamma_0 = m_1 + m_2$, where m_1 is the number of the simple zero of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$.

We consider the uniqueness problems of difference-differential polynomials $F^{(k)}(z)$ and obtained the following results, which improves the Theorems K and L.

Theorem 1.1. Let f be a transcendental meromorphic (resp. entire) function. n, k and m be positive integers with $n \geq k + 3 + \Gamma_0 - m$ (resp. $n \geq k + 2 + \Gamma_0 - m$) and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0, a_1, a_2, a_3, ..., a_{m-1}, a_m$

are complex constants and $\alpha(z)$ be a small function with respect to $f(z)$. Then $[f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)} - \alpha(z)$ has infinitely many zeros.

Remark 1.1. If $\Gamma_0 = m = m_1 + m_2$, $\lambda = 1$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem K.

Theorem 1.2. Let f and g be two non-constant entire functions of finite order. $F(z)$ and $G(z)$ are stated as in (1.1). Suppose that $n \geq 2k + 2\Gamma_0 + \lambda - m + 3$. If $F^{(k)}$ and $G^{(k)}$ share 1 CM, then

(1) when $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1(z) + a_0$, either $f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{n + \lambda_0 + \lambda, n + \lambda_1 + \lambda, ..., n + \lambda_m + \lambda\}$ and

$$\lambda_i = \begin{cases} i, & a_i \neq 0 \\ m, & a_i = 0 \end{cases} \qquad i = 0, 1, ..., m,$$

or f and g satisfy the algebraic equation $R(f,\, g) \equiv 0$, where

$$R(w_1,\, w_2) = w_1^n P(w_1)\prod_{j=1}^{d} w_1(z+c_j)^{s_j} - w_2^n P(w_2)\prod_{j=1}^{d} w_2(z+c_j)^{s_j}.$$

(2) when $P(z) \equiv c_0$ either $f(z) = \frac{c_1 e^{Cz}}{\sqrt[n]{c_0}}$, $g(z) = \frac{c_2 e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1, c_2, c_0 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+\lambda)C)^{2k} = (\sqrt[n]{c_0})^{2\lambda}$, or $f \equiv tg$ for a constant t such that $t^{n+\lambda} = 1$.

Remark 1.2. If $\Gamma_0 = m = m_1 + m_2$, $\lambda = 1$ in Theorem 1.2, then Theorem 1.2 improves the Theorem L.

2. Some Lemmas

For the proof of our main results, we need the following lemmas.

Lemma 2.1 ([1]). Let $f(z)$ be a transcendental meromorphic function of finite order, then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2 ([16]). Let $f(z)$ be a non-constant meromorphic function, and $a_n(\neq 0), a_{n-1}, ..., a_0$ be small functions with respect to f. Then

$$T(r, a_n f^n + a_{n-1}f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.3 ([6]). Let f be a transcendental meromorphic function of finite order. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

Lemma 2.4 ([7, 14]). Let $f(z)$ be a non-constant meromorphic function and $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

Lemma 2.5 ([16]). Let $f(z)$ and $g(z)$ be two transcendental entire functions, and k be a positive integer. Then

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.6 ([[1], [5]]). Let $f(z)$ be a meromorphic function of finite order and c is a non-zero complex constant. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.7 ([9], Lemma 2.3). Let $f(z)$ be a non-constant meromorphic function and p, k be positive integers. Then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.1}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.2}$$

Lemma 2.8. Let $f(z)$ be a transcendental entire function of finite order and let $F^* = f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}$. Then

$$T(r, F^*) = (n + m + \lambda)T(r, f) + S(r, f).$$

Proof. Since f is a transcendental entire function and also from Lemmas 2.2, 2.3 and 2.6, we obtain

$$
\begin{aligned}
(n+m+\lambda)T(r,f)+S(r,f) &= T(r,f^{n+\lambda}(z)P(f)) \le m(r,f^{n+\lambda}(z)P(f))+S(r,f)\\
&\le m\left(r,\frac{f^{\lambda}(z)F^*}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)+S(r,f)\\
&\le m(r,F^*)+S(r,f)\\
&\le T(r,F^*)+S(r,f).
\end{aligned}
$$

On the other hand, using Lemma 2.1 and f is a transcendental entire function of finite order, we have

$$
\begin{aligned}
T(r,F^*) &\le nT(r,f)+mT(r,f)+\lambda T(r,f(z+c))+S(r,f)\\
&\le (n+m+\lambda)T(r,f)+S(r,f).
\end{aligned}
$$

Hence we get Lemma 2.8.

3. Proof of the Theorems

Proof of Theorem 1.1. Denote $(F(z))^{(k)}=[f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ and $F=f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}$. From Lemma 2.8, F is not a constant. Next, we consider the following two cases.

Case 1. If f is a transcendental meromorphic function. Suppose that $F(z)^{(k)}-\alpha(z)$ has only finitely many zeros, then from the second fundamental theorem for three values and (2.1) of Lemma 2.7, we get

$$
\begin{aligned}
T(r,F^{(k)}) &\le \overline{N}(r,F^{(k)})+\overline{N}(r,0,F^{(k)})+\overline{N}(r,0,F^{(k)}-\alpha(z))+S(r,F^{(k)})\\
&\le \overline{N}(r,f)+N_1(r,0,F^{(k)})+\overline{N}(r,0,F^{(k)}-\alpha(z))+S(r,F^{(k)})\\
&\le \overline{N}(r,f)+T(r,F^{(k)})-T(r,F)+N_{k+1}(r,0,F)+S(r,F)+S(r,F^{(k)})
\end{aligned}
$$

(3.1)

$$T(r,F)\le \overline{N}(r,f)+N_{k+1}(r,0,F)+S(r,f).$$

From Lemma 2.8 and (3.1), it implies that

$$\begin{aligned}(n+m+\lambda)T(r,f)+S(r,f) &= T(r,F) \leq N_{k+1}(r,0,F)+\overline{N}(r,f)+S(r,f)\\ &\leq (k+1)\overline{N}(r,0,f)+N(r,0,P(f))+N\left(r,0,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)\\ &+\overline{N}(r,f)+S(r,f)\\ &\leq (k+2+\Gamma_0+\lambda)T(r,f)+S(r,f).\end{aligned}$$

Which is a contradiction to $n \geq k+3+\Gamma_0-m$. Hence $F(z)^{(k)}-\alpha(z)$ has infinitely many zeros.

Case 2. If f is a transcendental entire function. Suppose that $F(z)^{(k)}-\alpha(z)$ has only finitely many zeros. By using the same argument as in case 1, we get

$$(n+m+\lambda)T(r,f) \leq (k+1+\Gamma_0+\lambda)T(r,f)+S(r,f),$$

which is a contradiction to $n \geq k+2+\Gamma_0-m$. Hence $F(z)^{(k)}-\alpha(z)$ has infinitely many zeros.

Proof of Theorem 1.2.

(1) If $P(z)=a_m z^m+a_{m-1}z^{m-1}+...+a_1 z+a_0$.
Then by assumption and Theorem 1.1 we know that either both f and g are transcendental entire functions or both f and g are polynomials.
First, we consider the case when f and g are transcendental entire functions.
By Lemma 2.5, we have

$$T(r,F^{(k)}) \leq T(r,f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j})+S(r,f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}).$$

By Lemma 2.8, we get $S(r,F^{(k)})=S(r,f)$, similarly $S(r,G^{(k)})=S(r,g)$.
Since f, g are two transcendental entire functions with finite order, $F^{(k)}$ and $G^{(k)}$ share 1 CM, there exists a nonzero constant c such that

$$\frac{F^{(k)}-1}{G^{(k)}-1}=c.$$

Rewriting the above equation, we have

$$cG^{(k)}=F^{(k)}-1+c.$$

Assume that $c \neq 1$. By the second fundamental theorem and Lemma 2.7, we get

$$\begin{aligned} T(r, F^{(k)}) &\leq \overline{N}(r, F^{(k)}) + \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - 1 + c}\right) + S(r, F^{(k)}) \\ &\leq \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{G^{(k)}}\right) + S(r, f) \\ &\leq T(r, F^{(k)}) - T(r, F) + N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G^{(k)}}\right) + S(r, f) \\ &\leq T(r, F^{(k)}) - T(r, F) + N_{k+1}\left(r, \frac{1}{F}\right) + N_{k+1}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g). \end{aligned}$$

So

$$T(r, F) \leq N_{k+1}\left(r, \frac{1}{F}\right) + N_{k+1}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g).$$

By the definitions of F, G and Lemma 2.8, we have

$$(n + m + \lambda)T(r, f) \leq (k + 1 + \Gamma_0 + \lambda)[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

Similarly, we obtain

$$(n + m + \lambda)T(r, g) \leq (k + 1 + \Gamma_0 + \lambda)[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

Therefore

$$(n+m+\lambda)[T(r, f)+T(r, g)] \leq 2(k+1+\Gamma_0+\lambda)[T(r, f)+T(r, g)]+S(r, f)+S(r, g),$$

which contradicts with the assumption that $n \geq 2k + 2\Gamma_0 + \lambda - m + 3$. Hence $F^{(k)} \equiv G^{(k)}$.

By $(F(z))^{(k)} = (G(z))^{(k)}$, we get $F(z) = G(z) + Q(z)$, where $Q(z)$ is a polynomial of degree atmost $k - 1$. If $Q(z) \not\equiv 0$, then

$$\frac{f^n(z)P(f) \prod_{j=1}^{d} f(z + c_j)^{s_j}}{Q(z)} = \frac{g^n(z)P(g) \prod_{j=1}^{d} g(z + c_j)^{s_j}}{Q(z)} + 1.$$

By the second fundamental theorem and Lemma 2.8, we deduce that

$$
\begin{aligned}
(n+m+\lambda)T(r,f) &= T\left(r, \frac{f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}}{Q(z)}\right) + S(r,f) \\
&\leq \overline{N}\left(r, \frac{f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}}{Q(z)}\right) + \overline{N}\left(r, \frac{Q(z)}{f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) \\
&+ \overline{N}\left(r, \frac{Q(z)}{g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}}\right) + S(r,f) \\
&\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\
&+ \overline{N}\left(r, \frac{1}{P(g)}\right) + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} g(z+c_j)^{s_j}}\right) + S(r,f) + S(r,g) \\
&\leq (1+\Gamma_0+d)[T(r,f)+T(r,g)] + S(r,f) + S(r,g).
\end{aligned}
$$

Similarly, we obtain

$$(n+m+\lambda)T(r,g) \leq (1+\Gamma_0+d)[T(r,f)+T(r,g)] + S(r,f) + S(r,g).$$

So

$$(n+m+\lambda)[T(r,f)+T(r,g)] \leq 2(1+\Gamma_0+d)[T(r,f)+T(r,g)] + S(r,f) + S(r,g).$$

Which contradicts with the assumption that $n \geq 2k + 2\Gamma_0 + \lambda - m + 3$. Hence $Q(z) \equiv 0$. Then

$$f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j} = g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}. \tag{3.2}$$

Set $h = \frac{f}{g}$. If h is not a constant, from (3.2), we find that f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n P(w_1) \prod_{j=1}^{d} w_1(z + c_j)^{s_j} - w_2^n P(w_2) \prod_{j=1}^{d} w_2(z + c_j)^{s_j}.$$

If h is a constant. Substituting $f = gh$ into (3.2), we get

(3.3)
$$\prod_{j=1}^{d} g(z+c_j)^{s_j} [a_m g^{n+m}(h^{n+m+\lambda}-1)+a_{m-1}g^{n+m-1}(h^{n+m+\lambda-1}-1)+...+a_0 g^n(h^{n+\lambda}-1)] \equiv 0,$$

where $a_m(\neq 0), a_{m-1}, ...a_0$ are constants.

Since g is transcendental entire function, we have $\prod_{j=1}^{d} g(z + c_j)^{s_j} \not\equiv 0$. Then, from (3.3), we have

$$(3.4)\quad a_m g^{n+m}(h^{n+m+\lambda}-1)+a_{m-1}g^{n+m-1}(h^{n+m+\lambda-1}-1)+...+a_0 g^n(h^{n+\lambda}-1) \equiv 0.$$

If $a_m(\neq 0)$ and $a_{m-1} = a_{m-2} = ... = a_0 = 0$, then from (3.4) and g is transcendental function, we get $h^{n+m+\lambda} = 1$.

If $a_m(\neq 0)$ and there exists $a_i \neq 0\,(i \in \{0, 1, 2, ..., m-1\})$. Suppose that $h^{n+m+\lambda} \neq 1$, from (3.4), we have $T(r, g) = S(r, g)$ which is contradiction with transcendental function g. Then $h^{n+m+\lambda} = 1$. Similar to this discussion, we can see that $h^{n+j+\lambda} = 1$ when $a_j \neq 0$ for some $j = 0, 1, 2, ..., m$.

Thus, we have $f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{n + \lambda_0 + \lambda, n + \lambda_1 + \lambda, ..., n + \lambda_m + \lambda\}$ and $\lambda_i\,(i = 0, 1, 2, ..., m)$ is stated as in Theorem 1.2.

Now we consider the case when f and g are two polynomials.

By $[f^n(z)P(f) \prod_{j=1}^{d} f(z + c_j)^{s_j}]^{(k)}$ and $[g^n(z)P(g) \prod_{j=1}^{d} g(z + c_j)^{s_j}]^{(k)}$ share 1 CM, we have

(3.5)
$$[f^n(z)(a_m f^m+...+a_0) \prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}-1 = \beta\, [g^n(z)(a_m g^m+...+a_0) \prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}-1$$

where β is a non-zero constant. Let $deg\, f = l$, then by (3.5) we know that $deg\, g = l$.

Differentiating the two sides of (3.5), we get

$$(3.6)\qquad f^{n-k-\lambda}(z)\, q_1(z) = g^{n-k-\lambda}(z)\, q_2(z),$$

where $q_1(z)$, $q_2(z)$ are two polynomials with $deg\, q_1(z) = deg\, q_2(z) = (m+k+2\lambda)l - (k+1)$. By $n \geq 2k+2\Gamma_0+\lambda-m+3$, we get $deg\, f^{n-k-\lambda}(z) = (n-k-\lambda)l > deg\, q_2(z)$. Thus, by (3.6) we know that there exists z_0 such that $f(z_0) = g(z_0) = 0$.
Hence, by (3.5) and $f(z_0) = g(z_0) = 0$, we deduce that $\beta = 1$, that is,
(3.7)

$$[f^n(z)(a_m f^m + ... + a_0) \prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)} = [g^n(z)(a_m g^m + ... + a_0) \prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$$

Thus, we have
(3.8)

$$f^n(z)(a_m f^m + ... + a_0) \prod_{j=1}^{d} f(z+c_j)^{s_j} - g^n(z)(a_m g^m + ... + a_0) \prod_{j=1}^{d} g(z+c_j)^{s_j} = Q(z)$$

where $Q(z)$ is a polynomial of degree atmost $k-1$. Next we prove $Q(z) \equiv 0$. By rewriting (3.7) as

$$f^{n-k}(z)\, p_1(z) = g^{n-k}(z)\, p_2(z). \tag{3.9}$$

Where $p_1(z)$, $p_2(z)$ are two polynomials with $deg\, p_1(z) = deg\, p_2(z) = (m+k+\lambda)l - k$ and $deg\, f(z) = l$.
Hence total number of common zeros of $f^{n-k}(z)$ and $g^{n-k}(z)$ is atleast k. Thus, by (3.8) we deduce that $Q(z) \equiv 0$, that is
(3.10)

$$f^n(z)(a_m f^m + a_{m-1} f^{m-1} ... + a_0) \prod_{j=1}^{d} f(z+c_j)^{s_j} = g^n(z)(a_m g^m + a_{m-1} g^{m-1} ... + a_0) \prod_{j=1}^{d} g(z+c_j)^{s_j}.$$

Next, similar to the argument of (3.2), we get $f(z) \equiv tg(z)$ for a constant t such that $t^l = 1$, where $l = GCD\{n+\lambda_0+\lambda, n+\lambda_1+\lambda, ..., n+\lambda_m+\lambda\}$ and λ_i $(i = 0, 1, 2, ..., m)$ is stated as in Theorem 1.2.

(2) If $P(z) \equiv c_0$
By the assumption and Theorem 1.1, we know that either both f and g are transcendental entire functions or both f and g are polynomials.
First, we consider the case when both f and g are transcendental entire functions.
Let

$$F = f^n(z) c_0 \prod_{j=1}^{d} f(z+c_j)^{s_j},\ G = g^n(z) c_0 \prod_{j=1}^{d} g(z+c_j)^{s_j}$$

By the Theorem F and $n \geq 2k+2\Gamma_0+\lambda-m+3$, we obtain either $f(z) = \frac{c_1 e^{Cz}}{\sqrt[n]{c_0}}$, $g(z) = \frac{c_2 e^{-Cz}}{\sqrt[n]{c_0}}$, where c_1, c_2, c_0 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}((n+\lambda)C)^{2k} = (\sqrt[n]{c_0})^{2\lambda}$, or $f \equiv tg$ for a constant t such that $t^{n+\lambda} = 1$.

Now we consider the case when both f and g are two polynomials.
By $[f^n(z)c_0\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ and $[g^n(z)c_0\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$ share 1 CM, we have

$$(3.11) \qquad [f^n(z)c_0\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)} - 1 = \gamma \left[[g^n(z)c_0\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)} - 1\right].$$

Where γ is a non-zero constant. Let $deg\, f(z) = l$, then by (3.11) we know that $deg\, g(z) = l$. Differentiating the two sides of (3.11), we get

$$(3.12) \qquad f^{n-k-\lambda}(z)\, q_3(z) = g^{n-k-\lambda}(z)\, q_4(z)$$

where $q_3(z)$, $q_4(z)$ are two polynomials with $deg\, q_3(z) = deg\, q_4(z) = (k+2\lambda)l - (k+1)$. By $n \geq 2k+4$, we get $deg\, f^{n-k-\lambda}(z) = (n-k-\lambda)l > deg\, q_4(z)$.

Thus, by (3.12) we know that there exists z_0 such that $f(z_0) = g(z_0) = 0$.

Hence, by (3.11) and $f(z_0) = g(z_0) = 0$, we deduce that $\gamma = 1$, that is,

$$(3.13) \qquad [f^n(z)c_0\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)} = [g^n(z)c_0\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$$

Thus, we have

$$(3.14) \qquad f^n(z)\prod_{j=1}^{d} f(z+c_j)^{s_j} - g^n(z)\prod_{j=1}^{d} g(z+c_j)^{s_j} = Q_1(z)$$

where $Q_1(z)$ is a polynomial of degree atmost $k-1$. Next we prove $Q_1(z) \equiv 0$. By rewriting (3.13) as

$$(3.15) \qquad f^{n-k}(z)\, p_3(z) = g^{n-k}(z)\, p_4(z)$$

where $p_3(z)$, $p_4(z)$ are two polynomials with $deg\, p_3(z) = deg\, p_4(z) = (k+\lambda)l - k$ and $deg\, f(z) = l$.
Hence total number of common zeros of $f^{n-k}(z)$ and $g^{n-k}(z)$ is atleast k.
Thus, by (3.14) we deduce that $Q_1(z) \equiv 0$, that is,

$$(3.16) \qquad f^n(z)\prod_{j=1}^{d} f(z+c_j)^{s_j} = g^n(z)\prod_{j=1}^{d} g(z+c_j)^{s_j}.$$

Let $h(z) = \frac{f(z)}{g(z)}$ and $h(z+c) = \frac{f(z+c)}{g(z+c)}$ then

$$(gh)^n \prod_{j=1}^{d} g(z+c_j)^{s_j} h(z+c_j)^{s_j} = g^n \prod_{j=1}^{d} g(z+c_j)^{s_j}$$

Hence $f = tg$ where $(tg)^n \prod_{j=1}^{d} t(z+c_j)^{s_j} g(z+c_j)^{s_j} = g^n \prod_{j=1}^{d} g(z+c_j)^{s_j}$

$$\Rightarrow t^{n+\lambda} = 1.$$

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